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STRESS BASED YIELD/FAILURE CRITERIA FOR FIBER COMPOSITES

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Abstract — A macroscopic yield or failure criterion is derived for fiber composite materials. The derivation decomposes naturally into two modes of yield/failure, one being matrix dominated, the other being fiber dominated, thus there are two governing criteria. The resulting forms are quadratic in the components of the average stress tensor with two material parameters for each mode of yield failure. The physical context of the formulation is that of aligned fiber systems with a polymeric matrix phase under high fiber concentration conditions. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

In the present work a new theory of yielding and failure is derived for fiber composite materials. The importance of this general area is underscored by the high degree of activity that has been expended over the past 30 years or so in searching for the elusive "best" formulation. The difficulty of the search is evident from the multitude of empirical approaches that have been offered. The approach given here stresses the need to derive the governing forms from some basic physical postulations. In performing this derivation, a balance is sought between achieving the maximum generality and minimizing the number of materials parameters that must be experimentally evaluated for application to any particular material system.

No attempt is made here to survey the breadth of this failure criteria field for fiber composites. Rather, some of the prominent and typical contributions will be mentioned. Without any doubt, the best known and most widely applied failure criterion is that of Tsai and Wu (1971). The more primitive form of the Tsai-Wu criterion is that given by Hill (1950). Hashin (1980) presented a criterion that is completely different from the above forms. The aforementioned forms are all expressed in terms of stresses, and a simplified variant of these is the maximum stress form. In more recent years strain based formulations have been given, as by Christensen (1988) and Feng (1991). A simple variant of these is the maximum strain form, particularly when applied in the fiber direction. A recent useful evaluation of some of the many forms has been given by Hahn and Kallas (1992). Recent discussions of possible requirements in seeking failure criteria forms have been given by Hart-Smith (1993).

There are two key steps that guide the following derivation. In the first one, micromechanics is used to distinguish or discriminate different modes of failure. In particular, these are fiber dominated and matrix dominated modes. However, these two terms, fiber dominated and matrix dominated, are just convenient labels used to distinguish the two different modes of yield or failure that can occur in far different regions of stress space, different by as much as an order of magnitude or more. Of course, for both modes, both phases (fiber and matrix) play vital and strongly interactive roles.

The second key physical requirement is that of independence of hydrostatic compressive stress states, when imposed as a state by itself. That is, yield or failure is taken not to occur under independent states of hydrostatic compressive stress, within the range of the present second degree terms. This does not mean that yield and failure cannot occur beyond this range or under hydrostatic tensile stress states or under hydrostatic, meaning equi-normal, strain states. In the latter cases, the ensuing theory will indeed predict yield or failure.

Now the question arises as to which field variables to use in deriving the yield/failure criteria, stress based or strain based. The author does not know of any fundamental answer to this question, however, there is some evidence that suggests that stress is the preferred form. First, observe that the ideal (theoretical) compressive strength of composites, namely $\sigma_{11}^{C} = -\mu_L$ where μ_L is the longitudinal shear modulus, is naturally and simply expressed in terms of stress rather than strain.

Another facet suggests that stress is the preferred form. With metals, yielding is due to the flow of dislocations, which depends on shear stress or shear strain. The corresponding Mises criterion can equally well be expressed in terms of stresses or strains and is independent of mean normal stress or mean normal strain. In contrast, with polymeric materials there is ample evidence that yielding and failure are dependent upon the mean normal stress state. Now consider the conceptual limiting case of an incompressible isotropic material, meaning a material that cannot undergo volume change. A corresponding yield criterion expressed in terms of strains will be degenerate in equi-normal strain states whereas one expressed in terms of stresses remains well posed and useful. Therefore, in the present context of polymeric composites, it is expected that stress is a more appropriate yield variable than is strain, and stress will be used here.

From the previous discussion it is clear that the intended application is to polymeric composites, rather than metal matrix forms. Finally, within the usual convention of composite material terminology the terms yield and failure are used interchangeably, the distinction actually being in the particular manner of application to a particular material. For simplicity, from this point onward, the term yield will be used as covering both possible physical effects, yielding and failure. The objective is to determine yield criteria for aligned fiber composites in terms of the average, macroscopic stress variables.

MICROMECHANICS DISCRIMINATION

Although the following formulation is taken at a macroscopic level, guidance toward the proper forms can be obtained from some micromechanics considerations. In the present context, the micromechanics scale is that at which the fiber and matrix phases can be distinguished.

The fiber composite material on the macro-scale is taken to have transversely isotropic symmetry. For this class of symmetry, the second order stress tensor, σ_{ij}, has seven invariants through second degree polynomial terms. Taking rectangular cartesian axes with axis 1 in the fiber direction and that of the axis of symmetry, the seven invariants are

$$\sigma_{11},\ \sigma_{ii},\ \sigma_{11}^2,\ \sigma_{ii}^2,\ \sigma_{11}\sigma_{ii},\ \sigma_{1i}\sigma_{1i},\ \sigma_{ij}\sigma_{ij} \qquad i, j = 2, 3.$$

The existence of 2 first degree terms, σ_{11} and σ_{ii} $(i = 2, 3)$ suggests that there may be two modes of yield, especially since σ_{11} and σ_{ii} $(i = 2, 3)$ can differ by an order of magnitude or more. To pursue this matter further, it is useful to employ micromechanics.

Take the case of very stiff, meaning nearly rigid, fibers compared with the matrix phase. Interest in this work is in the high fiber loading case, and temporarily take $c = 0.7$ as the volume fraction of fibers. Using the Generalized Self Consistent Method, Christensen (1990), the effective transverse shear modulus, μ_T, vs Poisson's ratio of the matrix, ν_m is given by

ν_m	$\frac{\mu_T}{\mu_m}$
0.30	5.5031
0.32	5.6963
0.38	6.5905

where μ_m is the matrix phase shear modulus. The corresponding axial or longitudinal shear modulus, μ_L is given by

$$\frac{\mu_L}{\mu_m} = \frac{1+c}{1-c} = 5.667.$$

The corresponding axial Young's modulus, E_L, Hashin and Rosen (1964), Christensen (1990), is given by

$$E_L = cE_f + (1-c)E_m + \frac{4c(1-c)(v_f - v_m)^2 \mu_m}{(1-c)\left(\dfrac{\mu_m}{k_f + \dfrac{\mu_f}{3}}\right) + c\left(\dfrac{\mu_m}{k_m + \dfrac{\mu_m}{3}}\right) + 1}$$

where the various properties have an obvious identification. Under realistic conditions, this last expression is very well approximated by the first two terms, as

$$E_L \cong cE_f + (1-c)E_m$$

which is the rule of mixtures. The axial Young's modulus property and Poisson's ratio are, in fact, the only cases in which the rule of mixtures comes even close to giving the correct result obtained from any rigorous micromechanics model.

Switching to index notation, the above results are

Transverse shear	Axial shear	Axial extension
$\sigma_{23} = 13.2\ \mu_m \varepsilon_{23}$	$\sigma_{12} = 11.3\ \mu_m \varepsilon_{12}$	$\sigma_{11} = (0.7E_f + 0.3E_m)\,\varepsilon_{11}$

where ε_{ij} is the strain and the $v_m = 0.38$ case is used. Compare the three terms 13.2 μ_m, 11.3 μ_m and $0.3E_m$. The three numerical coefficients show that there is a very large stress concentration effect in the matrix phase in longitudinal or transverse shear cases, but not in longitudinal extension or contraction. These results are not peculiar to the Generalized Self Consistent Method, other micromechanics models show similar results.

In a relative sense there is a very energetic and non-uniform state in the matrix phase in axial or transverse shear (or transverse extension or contraction) whereas in the case of uni-axial stress in the fiber direction, there is a low energy state in the matrix. Consequently, it is appropriate to adopt the hypothesis that the state in the matrix due to macro-stress σ_{11} is of insufficient magnitude to interact with the state in the matrix due to the other components of macro-stress, even though macro-stress σ_{11} is very large due to the fiber contribution. Now use this matrix behavior to motivate the following broader point of view. Distinguish modes of yield that involve stress component σ_{11} from other modes of yield involving the other components of stress which themselves are not interactive with σ_{11} on the micro-scale.

In specific form, take two distinct modes of yield. In the matrix dominated mode all terms of stress except σ_{11} are interactive. Taking the above invariants for transverse isotropy except those involving σ_{11} gives the matrix dominated criterion as

Matrix dominated

$$f(\sigma_{ii}, \sigma_{ij}^2, \sigma_{1j}\sigma_{1i}, \sigma_{ij}\sigma_{ii}) \leqslant 1 \quad i, j = 2, 3 \tag{1}$$

where the inequality implies elastic behavior and the equality implies yielding. The complementary fiber dominated mode of yield then involves the remaining invariants for transverse isotropy as

Fiber dominated

$$F(\sigma_{11}, \sigma_{11}^2, \sigma_{11}\sigma_{ii}) \leqslant 1 \quad i = 2, 3. \tag{2}$$

As an alternate view from that of the micromechanics argument just given, one could appeal to the existence of extreme anisotropy as motivating the decomposition into modes (1) and (2), where σ_{11} is an order of magnitude larger than the other stress components.

It is clear than neither of the criteria (1) or (2) will individually admit the limiting form of a Mises criterion, since the decomposition has precluded that possibility. In the present context that is not a significant limitation since the entire motivation and derivation is aimed toward polymer composites of high fiber loading.

MODE I. MATRIX DOMINATED

Take the matrix dominated yield function form in (1) as expanded in a polynomial in the invariants up to terms of degree two, giving

$$\lambda\sigma_{ii} + \beta\sigma_{ii}^2 + \gamma\sigma_{ij}\sigma_{ij} + \Delta\sigma_{1i}\sigma_{1i} \leqslant 1 \quad i, j = 2, 3 \tag{3}$$

where λ, β, γ and Δ are material parameters.

Require that the form in (3) permit yielding under hydrostatic tensile stress, but not permit yielding under hydrostatic compressive stress. Take hydrostatic stress as

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{3D}. \tag{4}$$

Then, (3) becomes

$$2\lambda\sigma_{3D} + 2(2\beta + \gamma)\sigma_{3D}^2 \leqslant 1 \tag{5}$$

noting that σ_{11} is not involved. To be independent of hydrostatic compression, (5) requires

$$\beta = -\frac{\gamma}{2} \tag{6}$$

leaving

$$2\lambda\sigma_{3D} \leqslant 1. \tag{7}$$

Form (7) then possesses a tensile root, for λ non-negative. The range of validity of the physical assumption leading to (6) will be given in a later section.

Use restriction (6) and rescale the remaining parameters in (3) to give

$$\alpha k\sigma_{ii} + \frac{\gamma}{2}\left[-\frac{\sigma_{ii}^2}{2} + \sigma_{ij}\sigma_{ij}\right] + \sigma_{1i}\sigma_{1i} \leqslant k^2 \quad i, j = 2, 3 \tag{8}$$

where now the three parameters are α, γ and k, with the γ in (8) being changed by a factor of 2 from that in (3) for later convenience.

Take a state of one-dimensional transverse stress, σ_{22}, in (8), giving the form for the envelope as

$$\sigma_{22}^2 + \frac{4\alpha k}{\gamma}\sigma_{22} - \frac{4k^2}{\gamma} = 0. \tag{9}$$

This equation has the solution with two roots as

$$\sigma_{22}^{T} = \frac{2\alpha k}{\gamma}\left[-1+\sqrt{1+\frac{\gamma}{\alpha^2}}\right]$$

$$\sigma_{22}^{C} = \frac{-2\alpha k}{\gamma}\left[1+\sqrt{1+\frac{\gamma}{\alpha^2}}\right]. \tag{10}$$

The more general form of (8) including both σ_{22} and σ_{12} is

$$\alpha k \sigma_{22} + \frac{\gamma}{4}\sigma_{22}^2 + \sigma_{12}^2 \leqslant k^2. \tag{11}$$

At $\alpha = 0$ take $\gamma = 1$ and refer to this as the Mises-like special case, leaving (8) as

$$\frac{1}{2}\left(-\frac{\sigma_{ii}^2}{2} + \sigma_{ij}\sigma_{ij}\right) + \sigma_{1i}\sigma_{1i} \leqslant k^2 \quad i,j = 2,3. \tag{12}$$

This form is referred to as Mises-like since, in this special case, the yield in transverse shear equals the yield in longitudinal shear and the yield in transverse tension equals the yield in transverse compression. This form cannot be viewed as a purely Mises result since stress component σ_{11} is not involved, and thus the term Mises-like is used.

Now, in the general case, not the special Mises-like condition, take the yield in longitudinal shear and the yield in transverse compression to behave in accordance with Mises-like behavior, but allow the yield in transverse tension to be more severely degraded relative to the other two conditions. Thus, for σ_{12} by itself, at yield

$$\sigma_{12}^{Y} = k \tag{13}$$

and σ_{22}^{C} by itself, is taken as

$$\sigma_{22}^{C} = Ck \tag{14}$$

where C is some constant independent of α and γ. From the Mises-like condition (12), the constant C must be -2, and using (10) and (14)

$$\frac{-2\alpha k}{\gamma}\left[1+\sqrt{1+\frac{\gamma}{\alpha^2}}\right] = -2k. \tag{15}$$

The solution of (15) is simply

$$\gamma = 1+2\alpha. \tag{16}$$

Thus another material parameter has been eliminated using the condition stated ahead of eqns (13) and (14).

Using the preceding forms (10) and (11), and in particular (16), gives

$$\sigma_{22}^{C} = -2k$$

$$\sigma_{22}^{T} = \frac{2k}{1+2\alpha}$$

and

$$\sigma_{12}^{Y} = k. \tag{17}$$

The parameter $\alpha \geqslant 0$ in σ_{22}^{T} gives the physical condition that the yield in transverse tension is in general less than the yield magnitude in transverse compression.

The general matrix dominated yield criterion (8) with (16) takes the final form

$$\alpha k\sigma_{ii} + (\tfrac{1}{2}+\alpha)\left[\frac{-\sigma_{ii}^{2}}{2} + \sigma_{ij}\sigma_{ij}\right] + \sigma_{1i}\sigma_{1i} \leqslant k^{2} \quad i,j = 2,3 \tag{18}$$

where from (17)

$$\alpha = \frac{1}{2}\left(\frac{|\sigma_{22}^{C}|}{\sigma_{22}^{T}} - 1\right)$$

$$k = \sigma_{12}^{Y} = \frac{|\sigma_{22}^{C}|}{2}. \tag{19}$$

Parameters α and k later will be designated as α_1 and k_1 to distinguish this matrix dominated yield condition from the fiber dominated yield condition, to be derived next.

MODE II. FIBER DOMINATED

Take the fiber dominated yield criterion (2), with terms up to second degree as

$$\lambda\sigma_{11} + \beta\sigma_{11}^{2} + \gamma\sigma_{ii}\sigma_{11} \leqslant 1 \quad i = 2,3 \tag{20}$$

where λ, β and γ are the material parameters, appropriate to Mode II and not to be confused with like symbols in the previous section for Mode I.

With some guidance from the formalism of Mode I, rescale the forms in (20) to give

$$-\alpha k\sigma_{11} + \tfrac{1}{2}(\tfrac{1}{2}+\alpha)\sigma_{11}^{2} - \gamma\sigma\sigma_{11} \leqslant k^{2} \tag{21}$$

where

$$\sigma = \frac{\sigma_{22}+\sigma_{33}}{2}.$$

The materials parameters are now α, k and γ appropriate to the fiber dominated Mode II and different from the same symbols in Mode I or in (20). It is emphasized that in going from (20) to (21) no assumptions have been made and there remain three independent parameters at this stage. The form in (21) is beneficial for later interpretations.

Specializing (21) to the case of $\sigma = 0$, leaving only normal stress in the fiber direction, gives the two roots for σ_{11} as

$$\sigma_{11}^{T} = 2k$$

$$\sigma_{11}^{C} = \frac{-2k}{1+2\alpha}. \tag{22}$$

Thus the fiber direction yield stress in compression is degraded from that in tension by the factor $1+2\alpha$ where $\alpha \geqslant 0$.

Now, eliminate one of the three parameters in favor of the other two by using a special physical hypothesis. The same as was used in Mode I, here in the fiber dominated mode take the yield condition such that there is no yield under conditions of hydrostatic compression. With form (4) in (21), there is

$$[\tfrac{1}{3}(\tfrac{1}{2}+\alpha)-\gamma]\hat{\sigma}_{3D}^2-\alpha\hat{\sigma}_{3D} \leqslant 1 \tag{23}$$

where

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{k}.$$

The solution of the envelope in (23) is

$$\hat{\sigma}_{3D} = \frac{1}{(\frac{1}{2}+\alpha-2\gamma)}[\alpha \pm \sqrt{(1+\alpha)^2-4\gamma}]. \tag{24}$$

For

$$\gamma = \frac{(1+\alpha)^2}{4}$$

the solution from (24) is the double compression root

$$\hat{\sigma}_{3D} = -\frac{2}{\alpha}.$$

For

$$0 \leqslant \gamma < \frac{(1+\alpha)^2}{4}$$

there is a compressive root of (24). It is only in the case

$$\gamma > \frac{(1+\alpha)^2}{4} \tag{25}$$

that the roots of (24) are complex conjugate and there can be no yield under hydrostatic compression. In this case of (25) there is no second root either. Condition (25) will be used later.

Next determine the effect of an imposed hydrostatic stress upon the yield stress in the fiber direction, σ_{11}. Rewrite the yield condition (21) using the notation for $\hat{\sigma}_{ij}$ in (23). Then (21) is, for the envelope of yield,

$$\tfrac{1}{2}(\tfrac{1}{2}+\alpha)\hat{\sigma}_{11}^2-(\alpha+\gamma\hat{\sigma})\hat{\sigma}_{11}-1 = 0 \tag{26}$$

where now $\sigma = \sigma_{22} = \sigma_{33}$. Take the derivative of (26) with respect to $\hat{\sigma}$ and evaluate at $\hat{\sigma} = 0$. The result is

$$\left.\frac{\mathrm{d}\hat{\sigma}_{11}}{\mathrm{d}\hat{\sigma}}\right|_{\hat{\sigma}=0} = \frac{\gamma\hat{\sigma}_{11}}{(\frac{1}{2}+\alpha)\hat{\sigma}_{11}-\alpha}. \tag{27}$$

Evaluating the two roots using (22) gives

$$\left.\frac{d\hat{\sigma}_{11}^T}{d\hat{\sigma}}\right|_{\hat{\sigma}=0} = \frac{2\gamma}{(1+\alpha)}$$

and

$$\left.\frac{d\hat{\sigma}_{11}^C}{d\hat{\sigma}}\right|_{\hat{\sigma}=0} = \frac{2\gamma}{(1+\alpha)(1+2\alpha)}. \tag{28}$$

At $\alpha = 0$ (28) gives

$$\left.\frac{d\hat{\sigma}_{11}^T}{d\hat{\sigma}}\right|_{\hat{\sigma}=0} = \left.\frac{d\hat{\sigma}_{11}^C}{d\hat{\sigma}}\right|_{\hat{\sigma}=0} = 2\gamma|_{\alpha=0}. \tag{29}$$

There now is enough information to evaluate γ. In view of the inequality (25) take

$$\gamma = \frac{(1+\alpha)^2}{\eta} \tag{30}$$

and evaluate η to satisfy (25) and to give Mises-like behavior at $\alpha = 0$. At $\alpha = 0$, Mises-like behavior can be shown to be

$$\left.\frac{d\hat{\sigma}_{11}^T}{d\hat{\sigma}}\right|_{\hat{\sigma}=0} = \left.\frac{d\hat{\sigma}_{11}^C}{d\hat{\sigma}}\right|_{\hat{\sigma}=0} = 1 \quad (\alpha = 0). \tag{31}$$

Relations (31) simply express the Mises-like characteristic that the yield stress $\hat{\sigma}_{11}^T$ or $\hat{\sigma}_{11}^C$ under the action of a superimposed pressure increases only by the amount of the pressure. In other words the Mises-like yield is independent of the additional pressure. Evaluating η in (30) with (29) to satisfy (31) and (25) gives $\eta = 2$, leaving

$$\gamma = \frac{(1+\alpha)^2}{2}. \tag{32}$$

With (32), the derivatives (28) become

$$\left.\frac{d\hat{\sigma}_{11}^T}{d\hat{\sigma}}\right|_{\hat{\sigma}=0} = (1+\alpha)$$

and

$$\left.\frac{d\hat{\sigma}_{11}^C}{d\hat{\sigma}}\right|_{\hat{\sigma}=0} = \frac{(1+\alpha)}{(1+2\alpha)}. \tag{33}$$

Parameter γ evaluated in (32) gives the final form for the fiber dominated yield function (21) as

$$-\alpha k\sigma_{11} + \tfrac{1}{2}(\tfrac{1}{2}+\alpha)\sigma_{11}^2 - \frac{(1+\alpha)^2}{2}\sigma\sigma_{11} \leqslant k^2 \tag{34}$$

where

$$k = \frac{\sigma_{11}^{T}}{2}$$

$$\alpha = \frac{1}{2}\left(\frac{\sigma_{11}^{T}}{|\sigma_{11}^{C}|} - 1\right). \tag{35}$$

Material parameters k and α in (35) will be designated as k_2 and α_2 in the next section where both yield functions are assembled and interpreted.

COMBINED YIELD FAILURE CRITERIA

The criteria for the two modes of yield for the aligned fiber composite materials now will be composed together. Both modes of yielding must be considered in particular applications. The violation of either inequality must be taken as causing yielding and consequent damage in the composite. For some sub-spaces of stress, only one mode would be involved, while for others, both Modes I and II must be examined. In general, the overall yield surface will have limiting contributions from both Modes I and II.

From (18) and (19) the matrix dominated forms are

Mode I
Matrix Dominated

$$\alpha_1 k_1 \sigma_{ii} + (\tfrac{1}{2} + \alpha_1)\left[\frac{-\sigma_{ii}^2}{2} + \sigma_{ii}\sigma_{jj}\right] + \sigma_{1i}\sigma_{1i} \leqslant k_1^2 \quad i,j = 2,3 \tag{36}$$

or, in terms of specific components,

$$\alpha_1 k_1(\sigma_{22} + \sigma_{33}) + (1 + 2\alpha_1)[\tfrac{1}{4}(\sigma_{22} - \sigma_{33})^2 + \sigma_{23}^2] + (\sigma_{12}^2 + \sigma_{31}^2) \leqslant k_1^2 \tag{36a}$$

where

$$k_1 = \sigma_{12}^{Y} = \frac{|\sigma_{22}^{C}|}{2}$$

and

$$\alpha_1 = \frac{1}{2}\left(\frac{|\sigma_{22}^{C}|}{\sigma_{22}^{T}} - 1\right)$$

with σ_{12}^{Y}, σ_{22}^{T} and σ_{22}^{C} being the yield values in axial shear and transverse normal stress respectively.

From (34) and (35) in the fiber dominated case,

Mode II
Fiber Dominated

$$-\alpha_2 k_2 \sigma_{11} + \tfrac{1}{4}(1 + 2\alpha_2)\sigma_{11}^2 - \frac{(1 + \alpha_2)^2}{2}\sigma\sigma_{11} \leqslant k_2^2 \tag{37}$$

where

$$\sigma = \frac{\sigma_{22} + \sigma_{33}}{2}$$

$$k_2 = \frac{\sigma_{11}^T}{2}$$

and

$$\alpha_2 = \frac{1}{2}\left(\frac{\sigma_{11}^T}{|\sigma_{11}^C|} - 1\right)$$

with σ_{11}^T and σ_{11}^C being the fiber direction yields for $\sigma = 0$.

The restrictions that must accompany these two yield criteria are very simple. First, for the partition into two separate and distinct criteria, it is required that

$$k_1 \ll k_2. \tag{38}$$

The second restriction concerns the assumption that within the range of behavior considered here, yielding does not occur under a hydrostatic compressive state. It certainly remains open and likely that beyond this range, yield in hydrostatic compression could and would occur. The associated restriction then is that σ be of the order of k_1 and, with (38),

$$\left|\frac{\sigma_{22} + \sigma_{33}}{2}\right| = O(k_1) \ll k_2. \tag{39}$$

The complete description of the yield/failure criteria for fiber composites is given by (36)–(39). It is projected that these overall criteria by applicable to fiber composites under all stress conditions except those involving extreme hydrostatic pressure.

The yield criteria, (36) and (37) involve four materials parameters, α_1, k_1 and α_2, k_2. Parameters $k_1 > 0$ and $k_2 > 0$ are of the dimensions of stress and have a role similar to the corresponding parameter in a Mises criterion. Nondimensional parameters α_1 and α_2 impart a strongly non-Mises type of behavior to the yield functions. At $\alpha_1 = \alpha_2 = 0$ the two criteria have a Mises-like type of behavior with $|\sigma_{11}^C| = \sigma_{11}^T$ and $|\sigma_{22}^C| = \sigma_{22}^T$. Otherwise, non-zero positive values of α_1 and α_2 give a behavior of the type $|\sigma_{11}^C| < \sigma_{11}^T$ and $\sigma_{22}^T < |\sigma_{22}^C|$. Although $\alpha_1 \geqslant 0$ and $\alpha_2 \geqslant 0$, the normal range for these parameters would be from 0 to 1.

SPECIAL CASES

The yield function (36) for Mode I, matrix dominated, is shown in Fig. 1 for transverse normal stress, σ_{22} and axial shear stress σ_{12}. The yield envelopes are shown for $\alpha_1 = 0$ and $\alpha_1 = 1$. Other values of α_1 give a transition of shapes in between those shown in Fig. 1. It is noted from Fig. 1 and from (17) that at $\alpha_1 = 1$ the transverse tensile yield is 1/3 the value of the transverse compressive yield. Values of this ratio in the range of 1/2 to 1/3 are common.

Consider the relationship between σ_{22}^T, σ_{22}^C and σ_{12}^Y as shown in Fig. 1, and evaluate the relative magnitudes with specific data cases. The average of several data sets for graphite epoxy composites were given by Christensen and DeTeresa (1995). The strains at yield are

$$\varepsilon_{22}^T = 0.0060$$

$$\varepsilon_{22}^C = -0.019$$

$$\varepsilon_{12}^Y = 0.0056.$$

Using typical moduli of $E_{22} = 9$ GPa and $\mu_{12} = 5$ GPa, the corresponding stresses are

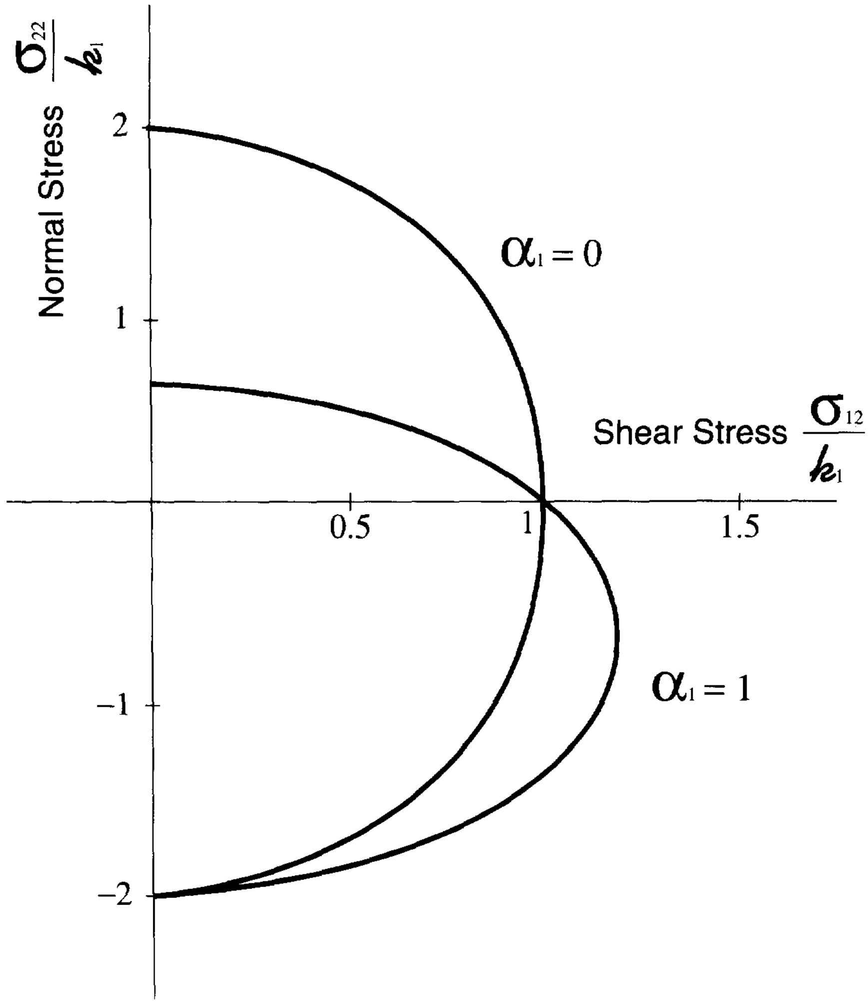

Fig. 1. Mode I, transverse normal stress vs axial shear stress.

$$\sigma_{22}^T = 54 \text{ MPa}$$

$$\sigma_{22}^C = -171 \text{ MPa}$$

$$\sigma_{12}^Y = 56 \text{ MPa}.$$

From (36) it follows that

$$\alpha_1 = 1.08.$$

The shear stress at yield and the transverse normal compressive stress at yield from the above data give

$$\frac{\sigma_{12}^Y}{|\sigma_{22}^C|} = 0.33$$

whereas from the yield function (36)

$$\frac{\sigma_{12}^Y}{|\sigma_{22}^C|} = \frac{1}{2}$$

for all values of α_1. The comparison between the last two values certainly is not close. This

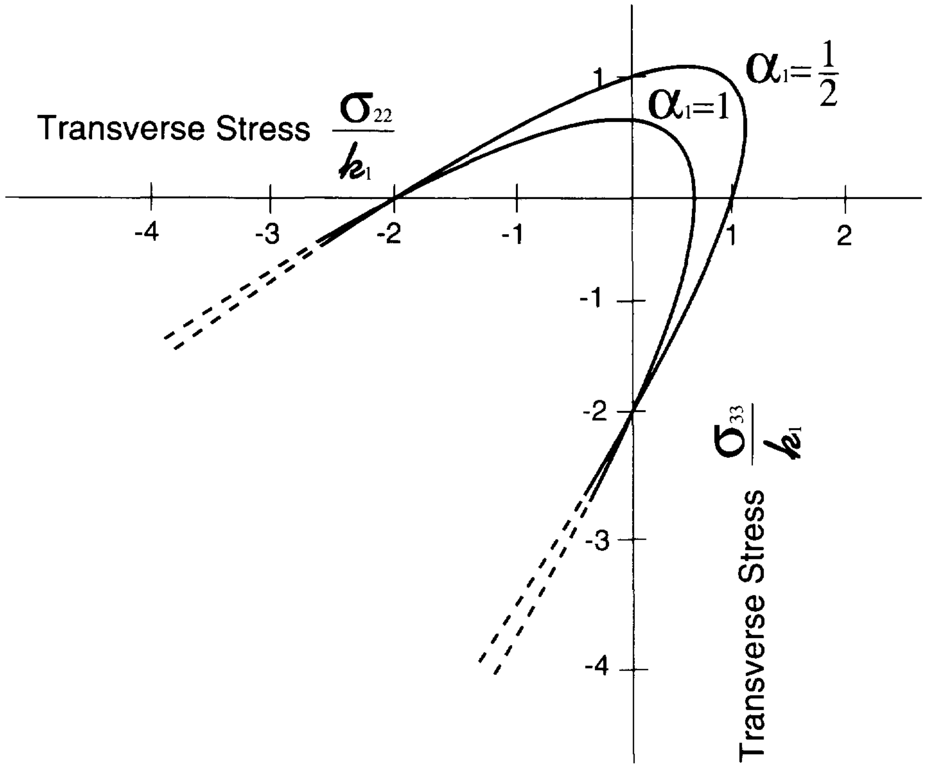

Fig. 2. Mode I, transverse normal stresses.

is largely due to the uncertainty in assigning a yield value to ε_{12}^Y in view of the extreme nonlinearity in the stress strain curve. Alternatively, if one directly uses the stresses at failure, then for typical data the above ratio appears to be much closer to 1/2. The consequence of the present theory that $|\sigma_{22}^C| = 2\sigma_{12}^Y$ under all conditions is probably not particularly significant compared with the many other unusual features of behavior for fiber composites. For example, a more meaningful evaluation probably is given by assessing the effect that pressure has upon σ_{12}^Y, as is done next.

Take the case of a pressure superimposed upon axial shear. With $\sigma_{22} = \sigma_{33} = \sigma$, the yield function (36) can be used to find the derivative

$$\left.\frac{d\sigma_{12}^Y}{d\sigma}\right|_{\sigma=0} = -\alpha_1 .$$

For $\alpha_1 = 1$, the value for this derivative equal to -1 is in the general range of behavior for graphite epoxy composites, showing that yield in axial shear increases strongly with pressure.

With respect to Mode I, the yield function (36) shows the relation between yield in transverse shear and yield in axial shear as

$$\sigma_{23}^Y = \frac{\sigma_{12}^Y}{\sqrt{1+2\alpha_1}} .$$

At $\alpha_1 = 1$ this gives $\sigma_{12}^Y = \sqrt{3}\sigma_{23}^Y$ and in general the yield in axial shear is larger than the yield in transverse shear.

Finally, for matrix dominated yield, the yield function (36) gives the behavior shown in Fig. 2 with respect to a σ_{22} vs σ_{33} subspace. Although the form in Fig. 2 appears to be open ended, the dotted portion is intended to indicate the effect of the restriction (39). At

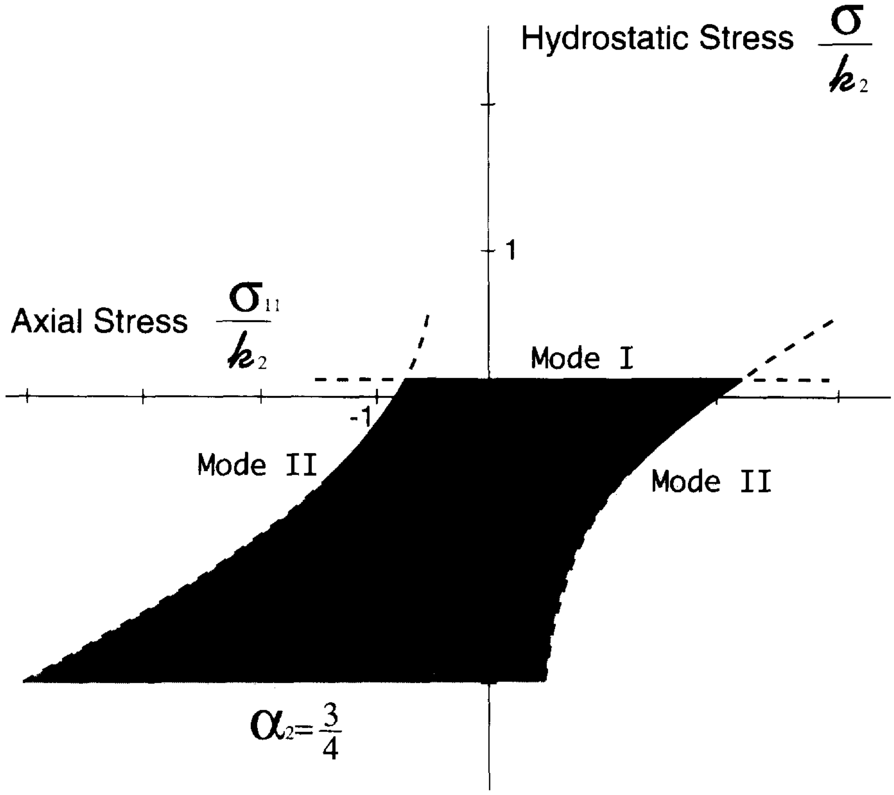

Fig. 3. Modes I and II, axial normal stress with hydrostatic stress, $\sigma = \sigma_{22} = \sigma_{33}$.

sufficiently large (extreme) pressures, the yield function in Fig. 2 would be expected to be closed. Also, as shown in Fig. 2, yield in hydrostatic tension can be very limiting.

One of the most critical tests of yield criteria for fiber composites is the effect that superimposed pressure has upon the yield in fiber direction stress. The slopes of the yield function crossing the σ_{11} axis were derived from the fiber dominated criterion (37) in the preceding section as relations (33), repeated here in slightly different form as

$$\left.\frac{d\sigma_{11}^T}{d\sigma}\right|_{\sigma=0} = (1+\alpha_2)$$

and

$$\left.\frac{d\sigma_{11}^C}{d\sigma}\right|_{\sigma=0} = \frac{(1+\alpha_2)}{(1+2\alpha_2)}. \tag{40}$$

These expressions will be evaluated but first the general characteristics of the fiber dominated yield function should be viewed. In Fig. 3 the general form of Mode II is given at $\alpha_2 = 3/4$. For $\alpha_2 = 3/4$, from (37) or (22)

$$\sigma_{11}^T = 2k_2$$

$$\sigma_{11}^C = -\tfrac{4}{5}k_2.$$

The ratio of these $(\sigma_{11}^T/|\sigma_{11}^C|) = 5/2$ is in the general range of properties for graphite epoxy composites. With $\alpha_2 = 3/4$, (40) becomes

eqn (40)	Experimental range
$\left.\dfrac{d\sigma_{11}^{T}}{d\sigma}\right\|_{\sigma=0} = 1.75$	(1.8–2.1)
$\left.\dfrac{d\sigma_{11}^{C}}{d\sigma}\right\|_{\sigma=0} = 0.7$	(0.58–0.62)

The experimental strength data are from Parry and Wronski (1982) and Parry and Wronski (1985), using the first four data points in their figures.

Returning to Fig. 3, the matrix dominated Mode I is also sketched in, in as much as it provides a limitation on the tensile values of σ. The exact position of this portion of the combined yield curve would depend upon the values of k_1 and α_1. Fitting the yield limits shown in Fig. 3 with a single expression would be extremely difficult. It also would be extremely unlikely that experimental data would precisely conform to the sharp cusp shape shown in Fig. 3 where Modes I and II join. Nevertheless, the general features of Fig. 3 are amenable to experimental examination.

The yield curves in Fig. 3 are shown as dotted in the larger magnitude negative range of σ. This is because the present second degree yield theory does not apply deeply into the large magnitude pressure range. A third degree theory would be needed under such extreme conditions, and that possibility will be explored in future work. Nevertheless, the present second degree theory may apply under most other broad and inclusive conditions. Finally, note that the Mode II yield form in Fig. 3 is not outwardly convex. A consequence of this is that the usual inviscid plasticity theory features are quite different from the yielding and/or failure characterizations derived here. The outwardly concave nature of the Mode II yield form implies that the resulting failure is due to an instability.

There is little doubt that some other failure theories involving more adjustable parameters than the four involved here could be "fitted" to larger sets of data than would be possible here. The question comes down to this: what are the major aspects of behavior and what are the minor ones? For example, the present theory has no coupling between fiber direction stress, σ_{11}, and axial shear stress σ_{12} or σ_{31}. A weak coupling between these would certainly suggest an idealization of no coupling. In contrast to this scenario, consider the effect of pressure upon the tensile and compressive yield of fiber direction stress, σ_{11}. This is well understood to be a major effect. Compared with the experimental data of Parry and Wronski the present theory reveals a credible prediction of the effect of pressure upon the yield of both the tensile and compressive stress in the fiber direction. Some other theories with more parameters not only do not predict the correct sizes for these physical effects, they do not even predict the correct sign. The strong diminishment of σ_{11}^{T} due to superimposed pressure could be especially important in design situations. In all matters of this type, there is a major opportunity for critical experimental data generation and for a critical evaluation of the means of damage characterization.

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REFERENCES

Christensen, R. M. (1988). Tensor transformations and failure criteria for the analysis of fiber composite materials. *J. Comp. Mat.* **22**, 874–897.

Christensen, R. M. (1990). A critical evaluation for a class of micromechanics models. *J. Mech. Phys. Solids* **38**, 379–404.

Christensen, R. M. and DeTeresa, S. J. (1996). The kink band mechanism for the compressive failure of fiber composite materials. *J. Appl. Mech.* (in press).

Feng, W. W. (1991). A failure criterion for composite materials. *J. Comp. Mat.* **25**, 88–100.

Hahn, H. T. and Kallas, M. N. (1992). Failure criteria for thick composites. Ballistic Research Laboratory Report BRL-CR-691, Aberdeen, MD.

Hart-Smith, L. J. (1993). Should fibrous composites failure modes be interacted or superimposed. *Composites* **24**, 53–55.

Hashin, Z. (1980). Failure criteria for unidirectional fiber composites. *J. Appl. Mech.* **47**, 329–334.

Hashin, Z. and Rosen, B. W. (1964). The elastic moduli of fiber-reinforced materials. *J. Appl. Mech.* **31**, 223–232.

Hill, R. (1950). *The Mathematical Theory of Plasticity*, Clarendon Press, Oxford.

Parry, T. V. and Wronski, A. S. (1982) Kinking and compressive failure in uniaxially aligned carbon fiber composite tested under superimposed hydrostatic pressure. *J. Mat. Sci.* **17**, 893–900.

Parry, T. V. and Wronski, A. S. (1985). The effect of hydrostatic pressure on the tensile properties of pultruded CFRP. *J. Mat. Sci.* **20**, 2141–2147.

Tsai, S. W. and Wu, E. M. (1971). A general theory of strength for anisotropic materials. *J. Comp. Mat.* **5**, 58–80.